OBJECTIVE MATHEMATICS

Volume 2

Descriptive Test Series

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CHAPTER-1 : FUNCTIONS

UNIT TEST-1

1. If the function $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = |x|(x - \sin x)$, then which of the following statements is **TRUE**?

(a) f is one-one, but **NOT** onto
(b) f is onto, but **NOT** one-one
(c) f is **BOTH** one-one and onto
(d) f is **NEITHER** one-one **NOR** onto

2. Let $f : \mathbf{R} \to \mathbf{R}$ satisfy $f(x + y) = 2^x f(y) + 4y f(x)$, $\forall x, y \in \mathbf{R}$. If $f(2) = 3$, then $14 \cdot \dfrac{f'(4)}{f'(2)}$ is equal to __

3. Let $f : \mathbf{R} \to \mathbf{R}$ be a function defined by

$$f(x) = \frac{2e^{2x}}{e^{2x} + e}.$$

Then $f\left(\frac{1}{100}\right) + f\left(\frac{2}{100}\right) + f\left(\frac{3}{100}\right) + \ldots + f\left(\frac{99}{100}\right)$ is equal to_____.

4. Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is the number of one-one functions from X to Y and β is the number of onto functions from Y to X, then the value of $\frac{1}{5!}(\beta - \alpha)$ is____.

5. If the domain of the function $f(x) = \sec^{-1}\left(\frac{2x}{5x+3}\right)$ is $[\alpha, \beta) \cup (\gamma, \delta]$, then $|3\alpha + 10(\beta + \gamma) + 21\delta|$ is equal to ______.

6. Let $R = \{a, b, c, d, e\}$ and $S = \{1, 2, 3, 4\}$. Total number of onto functions $f : R \to S$ such that $f(a) \neq 1$, is equal to _______.

7. If domain of the function

$$\log_e\left(\frac{6x^2 + 5x + 1}{2x - 1}\right) + \cos^{-1}\left(\frac{2x^2 - 3x + 4}{3x - 5}\right) \text{ is } (\alpha, \beta)$$

$\cup (\gamma, \delta]$, then $18(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$ is equal to _________.

Hints and Solutions

1. (c) $f(x)$ is a non-periodic, continuous and odd function

$$f(x)\begin{cases} -x + x\sin x, x < 0 \\ x - x\sin x, x \geq 0 \end{cases}$$

$$f'(x) = \begin{cases} -2x + \sin x + x\cos x, x < 0 \\ 2x - \sin x - x\cos x, x \geq 0 \end{cases}$$

$$f'(x) = \begin{cases} -(x - \sin x) - x(1 - \cos x), x < 0 \\ (x - \sin x) + x(1 - \cos x), x > 0 \end{cases}$$

$\therefore$ $x - \sin x < 0$ if $x < 0$ and

$1 - \cos x > 0, \forall x \in R$

$\therefore$ $-(x - \sin x) - x(1 - \cos x) > 0$ if $x < 0$

and $(x - \sin x) + x(1 - \cos x) > 0$ if $x > 0$

$\Rightarrow f'(x) > 0 \;\forall\; x \in \mathrm{R} \Rightarrow f(x)$ is increasing in R

$\Rightarrow f(x)$ is one-one

$$\therefore \operatorname*{Lt}_{x\to-\infty}(-x^2)\left(1 - \frac{\sin x}{x}\right) = -\infty \;\therefore \operatorname*{Lt}_{x\to\infty} x^2\left(1 - \frac{\sin x}{x}\right) = \infty$$

$\Rightarrow$ Range of $f(x) = \mathrm{R} \Rightarrow f(x)$ is an onto function

2. (248) $f(x + y) = 2^x f(y) + 4^y f(x)$ ... (1)

Now, $f(y + x) 2^y f(x) + 4^x f(y)$ ... (2)

$\therefore\ 2^x f(y)+4^y f(x)=2^y f(x)+4^x f(y)$

$\left(4^y-2^y\right)f(x)=\left(4^x-2^x\right)f(y)$

$\dfrac{f(x)}{4^x-2^x}=\dfrac{f(y)}{4^y-2^y}=k$ (Say)

$\therefore\ f(x)=k\left(4^x-2^x\right)$

$\because f(2)=3$ then $k=\dfrac{1}{4}$

$\therefore\ f(x)=\dfrac{4^x-2^x}{4}$

$\therefore f'(x)=\dfrac{4^x\ln 4-2^x\ln 2}{4}$

$f'(x)=\dfrac{\left(2.4^x-2^x\right)\ \ln 2}{4}$

$\therefore\ \dfrac{f'(4)}{f'(2)}=\dfrac{2.256-16}{2.16-4}$

$\therefore 14\,\dfrac{f'(4)}{f'(2)}=248$

3. (99) $f(x)=\dfrac{2e^{2x}}{e^{2x}+e^x}$ and $f(1-x)=\dfrac{2e^{2-2x}}{e^{2-2x}+e^{1-x}}$

$\therefore\ \dfrac{f(x)+f(1-x)}{2}=1$

i.e. $f(x)+f(1-x)=2$

$\therefore\ f\left(\dfrac{1}{100}\right)+f\left(\dfrac{2}{100}\right)+\ldots+f\left(\dfrac{99}{100}\right)$

$=\ \sum_{x=1}^{49} f\left(\dfrac{x}{100}\right)+f\left(1-\dfrac{x}{100}\right)+f\left(\dfrac{1}{2}\right)$

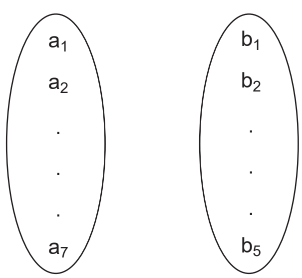
$=\ 49\times 2+1=99$

4. (119) Here $n(x)=5$ and $n(Y)=7$

Number of one-one function

$=\alpha={}^7C_5\times 5\,!$

and Number of onto function Y to $X=\beta$



1,1,1,1,3 1,1,1,2,2

$=\dfrac{7!}{3!4!}\times 5!+\dfrac{7!}{(2!)^3 3!}\times 5!=\left({}^7C_3+3\times{}^7C_3\right)5!$

$=4\times{}^7C_3\times 5!$

$\Rightarrow \dfrac{\beta-\alpha}{5!}=4\times{}^7C_3-{}^7C_5=4\times 35-21=119$

5. (24.00) $\dfrac{2x}{5x+3}\geq 1$ OR ≤ -1

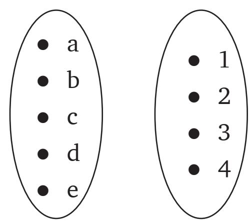
$\dfrac{-3x-3}{5x+3}\geq 0 \qquad \dfrac{7x+3}{5x+3}\leq 0$

$\dfrac{x+1}{5x+3}\leq 0 \qquad x\in\left(\dfrac{-3}{5},\dfrac{-3}{7}\right]$

$x\in\left[-1,-\dfrac{3}{5}\right)$

$3\alpha=-3,\ 10(\beta+\gamma)=-12,\ 21\delta=-9$

6. (180*) If $f(a)=1$



If $f(a)=1$ & one of $f(b), f(c), f(d), f(e)=1$ then total such cases $=4\cdot 3!=24$

If only $f(a)=1$, then

Total cases $=3^4-({}^3C_1\cdot 2^4)+({}^3C_2\cdot 1)=36$

No. of onto functions when $f(a)=1$ is $24+36=60$

Total no. of onto functions

$=4^5-({}^4C_1\cdot 3^5)+({}^4C_2\cdot 2^5)-({}^4C_3\cdot 1)$

$=1024-973+192-4=240$

Number of required functions $=240-60=180$

7. (20) $\dfrac{6x^2+5x+1}{2x-1}>0$ ...(1)

$-1\leq\dfrac{2x^2-3x+4}{3x-5}\leq 1$ ...(2)

From (1)

$\dfrac{6x^2+5x+1}{2x-1}>0 \quad\Rightarrow\quad \dfrac{(3x+1)(2x+1)}{(2x-1)}>0$

$\Rightarrow x\in\left(-\dfrac{1}{2},-\dfrac{1}{3}\right)\cup\left(\dfrac{1}{2},\infty\right)$ ...(a)

From (2)

$-1\leq\dfrac{2x^2-3x+4}{3x-5}\leq 1\Rightarrow\dfrac{2x^2-3x+4}{3x-5}\geq -1$ ...(3)

$$\text{and } \frac{2x^2-3x+4}{3x-5} \le 1 \qquad \ldots(4)$$

From (3)

$$\frac{2x^2-3x+4}{3x-5}+1 \ge 0 \Rightarrow x \in \left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \cup \left(\frac{5}{3}, \infty\right) \qquad \ldots(b)$$

From (4)

$$\frac{2x^2-3x+4}{3x-5}-1 \le 0 \Rightarrow \frac{2x^2-6x+9}{3x-5} \le 0$$

$$\Rightarrow \frac{1}{3x-5} \le 0 \quad \left(\because 2x^2-6x+9>0 \ \forall\ x \in R\right)$$

$$\Rightarrow x \in \left(-\infty, \frac{5}{3}\right) \qquad \ldots(c)$$

$\Rightarrow$ Intersection of (a), (b) and (c)

$$\left(-\frac{1}{2}, -\frac{1}{3}\right) \cup \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right]$$

$$\Rightarrow \alpha^2+\beta^2+\gamma^2+\delta^2 = \frac{10}{9}$$

$$\Rightarrow 18\left(\alpha^2+\beta^2+\gamma^2+\delta^2\right) = 20$$