OBJECTIVE MATHEMATICS

Volume 2

Descriptive Test Series

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CHAPTER-13 : DETERMINANTS

UNIT TEST-1

1. Let $D_k = \begin{vmatrix} 1 & 2k & 2k-1 \\ n & n^2+n+2 & n^2 \\ n & n^2+n & n^2+n+2 \end{vmatrix}$. If $\sum_{k=1}^{n} D_k = 96$, then n is equal to ______.

2. The number of matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c, $d \in \{-1, 0, 1, 2, 3\text{.....}, 10\}$, such that $A = A^{-1}$ is ______.

3. Let p and $p + 2$ be prime numbers and let

$$\Delta = \begin{vmatrix} p! & (p+1)! & (p+2)! \\ (p+1)! & (p+2)! & (p+3)! \\ (p+2)! & (p+3)! & (p+4)! \end{vmatrix}$$

Then the sum of the maximum values of α and β, such that p^α and $(p + 2)^\beta$ divide Δ, is ______.

4. Consider a matrix $A = \begin{bmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{bmatrix}$, where α, β, γ are three distinct natural numbers. If $\dfrac{\det(\text{adj}(\text{adj}(\text{adj}(\text{adj } A))))}{(\alpha-\beta)^{16}(\beta-\gamma)^{16}(\gamma+\alpha)^{16}} = 2^{32} \times 3^{16}$, then the number of such 3-tuples (α, β, γ) is ______.

5. The system of equations $2x + y = 4$, $3x + 2y = 2$, $x + y = -2$, has ______.

(a) Infinitely many solution
(b) No solution
(c) One solution
(d) Only 2 solutions

6. For real numbers α and β consider the following system of linear equations :
$x + y - z = 2$, $x + 2y + \alpha z = 1$, $2x - y + z = \beta$.
If the system has infinite solutions, then $\alpha + \beta$ is equal to ______.

7. If the following system of linear equations

$$2x + y + z = 5$$
$$x - y + z = 3$$
$$x + y + az = b$$

has no solution, then :

(a) $a = -\dfrac{1}{3}, b \neq \dfrac{7}{3}$ (b) $a \neq -\dfrac{1}{3}, b = \dfrac{7}{3}$

(c) $a \neq \dfrac{1}{3}, b = \dfrac{7}{3}$ (d) $a = \dfrac{1}{3}, b \neq \dfrac{7}{3}$

Hints and Solutions

1. (d) $\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & a \end{vmatrix} = 1 - 3a$

$$\Delta_3 = \begin{vmatrix} 2 & 1 & 5 \\ 1 & -1 & 3 \\ 1 & 1 & b \end{vmatrix} = 7 - 3b$$

For no solution, $\Delta = 0, \Delta_3 \neq 0$

$1 - 3a = 0$, $7 - 3b \neq 0$

$$a = \frac{1}{3}, b \neq \frac{7}{3}$$

2. (c) We have three equations:

$2x + y = 4$, $3x + 2y = 2$, $x + y = -2$

Subtracting first from second equation, we get
$x + y = -2$, which is same as the third equation.

Therefore third equation is redundant.

Solving first and second equation, we get
$x = 6, y = -8$

Hence, there is only one solution for the system of equations.

3. (5) $\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & \alpha \\ 2 & -1 & 1 \end{vmatrix} = 0 \Rightarrow \alpha = -2$

$$\Delta_2 = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & -2 \\ \beta & -1 & 1 \end{vmatrix} = 0 \Rightarrow \beta = 7$$

$$\Delta_3 = 0 \Rightarrow \beta = 7$$

$$\alpha + \beta = 5$$

4. (6) $\sum_{k=1}^{n} D_k = \begin{vmatrix} \sum 1 & 2\sum k & 2\sum k - \sum 1 \\ n & n^2+n+2 & n^2 \\ n & n^2+n & n^2+n+2 \end{vmatrix}$

$$= \begin{vmatrix} n & n(n+1) & n^2 \\ n & n^2+n+2 & n^2 \\ n & n^2+n & n^2+n+2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -2 & 0 \\ 0 & 2 & -n-2 \\ n & n^2+n & n^2+n+2 \end{vmatrix}$$

$$= 2((-n)\ (-n-2))$$

$$= 96$$

$$n^2 + 2n = 48$$

$$n = 6, -8$$

$$\boxed{n = 6}$$

5. (50) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$A^2 = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{bmatrix}$$

For A^{-1} must exist $ad - bc \neq 0$ ...(i)

and $A = A^{-1}$

$\Rightarrow$ $A^2 = 1$

$\therefore$ $a^2 + bc = d^2 + bc = 1$ ...(ii)

and $b(a + d) = c(a + d) = 0$ ...(iii)

Case I : When $a = d = 0$, then possible values of (b, c) are $(1, 1)$, $(-1, 1)$ and $(1, -1)$ and $(-1, 1)$.

Total four matrices are possible.

Case II : When $a = -d$ then (a, d) be $(1, -1)$ or $(-1, 1)$.

Then total possible values of (b, c) are

$$(12 + 11) \times 2 = 46.$$

$\therefore$ Total possible matrices $= 46 + 4 = 50$.

6. (4)

$$\Delta = \begin{vmatrix} p! & (p+1)! & (p+2)! \\ (p+1)! & (p+2)! & (p+3)! \\ (p+2)! & (p+3)! & (p+4)! \end{vmatrix}$$

$$= p!\cdot(p+1)!\cdot(p+2)! \begin{vmatrix} 1 & (p+1) & (p+1)(p+2) \\ 1 & (p+2) & (p+2)(p+3) \\ 1 & (p+3) & (p+3)(p+4) \end{vmatrix}$$

$$= p!\cdot(p+1)!\cdot(p+2)! \begin{vmatrix} 1 & p+1 & p^2+3p+2 \\ 0 & 1 & 2p+4 \\ 0 & 1 & 2p+6 \end{vmatrix}$$

$= 2(p!)\cdot((p + 1)!)\cdot((p + 2)!)$.

$= 2(p + 1)\cdot(p!)^2\cdot((p + 2)!)$.

$= 2(p + 1)^2\cdot(p!)^3\cdot((p + 2)!)$.

$\therefore$ Maximum value of α is 3 and β is 1.

$\therefore$ $\alpha + \beta = 4$

7. (42)

$$\det(A) = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} \qquad R_3 \,!\, R_3 + R_1$$

$$\Rightarrow (\alpha+\beta+\gamma) \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$\therefore$ $\det(A) = (\alpha + \beta + \gamma)\ (\alpha - \beta)\ (\beta - \gamma)\ (\gamma - \alpha)$

Also, det (adj (adj (adj (adj (A)))))

$$= (\det(A))^{2^4} = (\det(A))^{16}$$

$$\therefore \frac{(\alpha+\beta+\gamma)^{16}\,(\alpha-\beta)^{16}\,(\beta-\gamma)^{16}\,(\gamma-\alpha)^{16}}{(\alpha-\beta)^{16}\,(\beta-\gamma)\,(\gamma-\alpha)^{16}} = (4.3)^{16}$$

$\therefore$ $\alpha + \beta + \gamma = 12$

$\therefore$ (α, β, γ) distinct natural triplets

$= {}^{11}C_2 - 1 - {}^{3}C_2\ (4) = 55 - 1 - 12 = 42$