OBJECTIVE MATHEMATICS

Volume 1

Descriptive Test Series

Prof. M. L. Khanna
Bhushan Muley

CHAPTER-4 : PROGRESSIONS

UNIT TEST-1

1. For $0 < \theta < \frac{\pi}{4}$, let $S(\theta) = 1 + (1 + \sin\theta)\cos\theta + (1 + \sin\theta + \sin^2\theta)\cos^2\theta + \ldots\infty$.

Then find the value of $\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) S\left(\frac{\pi}{4}\right)$.

2. If $\tan\left(\frac{\pi}{12} - x\right)$, $\tan\frac{\pi}{12}$, $\tan\left(\frac{x}{12} + x\right)$ in order are three consecutive terms of a G.P., then sum of all the solutions in [0, 314] is $k\pi$. Find the value of k.

3. If a, b, c, d, e be 5 numbers such that a, b, c are in AP, b, c, d in G.P. and c, d, e are in H.P. then,

(i) Prove that a, c, e are in GP

(ii) Prove that $e = \frac{(2b-a)^2}{a}$

(iii) If $a = 2$ and $e = 18$, find all possible values of b, c, d.

4. Two distinct, real infinite geometric series each have a sum of 1 and have the same second term. The third term of one of the series is 18. If the second term of both the series can be written in the form $\frac{m-n}{p}$, where m, n and p are positive integers, and m is not divisible by the square of any prime, find the value of $100\,m + 10\,n + p$.

5. If $a_1, a_2, a_3, \ldots$ are in A.P. such that $a_i \neq 0$, show that $S = \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \ldots + \frac{1}{a_n a_{n+1}} = \frac{n}{a_1 a_{n+1}}$

Also evaluate $\lim\limits_{a \to \infty} S$.

6. Let $S_n = \sum\limits_{k=1}^{n} \frac{n}{n^2 + kn + k^2}$ and $T_n = \sum\limits_{k=0}^{n-1} \frac{n}{n^2 + kn + k^2}$ for n = 1, 2, 3, ... The,

(a) $S_n < \frac{\pi}{3\sqrt{3}}$ (b) $S_n > \frac{\pi}{3\sqrt{3}}$

(c) $T_n < \frac{\pi}{3\sqrt{3}}$ (d) $T_n > \frac{\pi}{3\sqrt{3}}$

7. Let $S_k, k = 1, 2, \ldots, 100.$ denotes the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$. Then the value of $\frac{100^2}{100!} + \sum\limits_{k=1}^{100} |(k^2 - 3k + 1)S_k|$ is

8. Find a three digit number whose consecutive numbers form a G.P. If we subtract 792 from this number, we get a number consisting of the same digits written in the reverse order. Now if we increase the second digit of the required number by 2, the resulting number will form an A.P.

9. If S be the sum, P the product and R the sum of the reciprocals of n terms is a G.P., prove that $P^2 = \left(\frac{S}{R}\right)^n$

10. Given that $a_1, a_2, a_3, \ldots, a_n$ form an A.P. find the following sum

$$S = \sum_{i=1}^{n} \frac{a_i a_{i+1} a_{i+2}}{a_i + a_{i+2}}$$

Hints and Solutions

1. $S(\theta) = 1 + (1 + \sin\theta)\cos\theta + (1 + \sin\theta + \sin^2\theta)\cos^2\theta \ldots\infty$

$= 1 + \cos\theta + \cos^2\theta \ldots + \sin\theta(\cos\theta + \cos^2\theta\ldots) + \sin^2\theta$

$$= \frac{1}{1-\cos\theta} + \frac{\sin\theta}{1-\cos\theta}\cos\theta + \frac{\sin^2\theta\cos^2\theta}{1-\cos\theta}$$

$$= \frac{1}{1-\cos\theta}[1 + \sin\theta\cos\theta + \sin^2\theta\cos^2\theta\ldots]$$

$$S(\theta) = \frac{1}{(1-\sin\theta\cos\theta)(1-\cos\theta)}$$

$$S_{\left(\frac{\pi}{4}\right)} = \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{\sqrt{2}}\right)} = \frac{2\sqrt{2}}{(\sqrt{2}-1)}$$

Solutions are $0, \pi, 2\pi, 3\pi \ldots 99\pi$

$$\frac{99}{2}[2\pi + 98\pi] = 50\pi \cdot 99 = 4950\pi$$

$$K = 4950$$

2. $\tan\left(\frac{\pi}{12} - x\right), \tan\frac{\pi}{12}, \tan\left(\frac{\pi}{12} + x\right)$ are in GP

$$\tan^2\frac{\pi}{12} = \tan\left(\frac{\pi}{12} - x\right)\tan\left(\frac{\pi}{12} + x\right)$$

$$= \frac{\sin\left(\frac{\pi}{12} + x\right)\sin\left(\frac{\pi}{12} - x\right)}{\cos\left(\frac{\pi}{12} + x\right)\cos\left(\frac{\pi}{12} - x\right)}$$

$$\Rightarrow \quad \frac{\cos 2x - \cos\frac{\pi}{6}}{\cos 2x + \cos\frac{\pi}{6}} = \tan^2\frac{\pi}{12}$$

$$\cos 2x = \frac{\cos\frac{\pi}{6}\tan^2\frac{\pi}{12} + \cos\frac{\pi}{6}}{1 - \tan^2\frac{\pi}{12}}$$

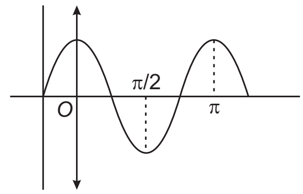
$$= \frac{\cos\frac{\pi}{6}\left[\tan^2\frac{\pi}{12} + 1\right]}{1 - \tan^2\frac{\pi}{12}}$$

$$= \cos\frac{\pi}{6}\left[\frac{\sin^2\left(\frac{\pi}{12}\right) + \cos^2\left(\frac{\pi}{12}\right)}{\cos^2\left(\frac{\pi}{12}\right) - \sin^2\left(\frac{\pi}{12}\right)}\right]$$

$$= \frac{\sqrt{3}}{2}\left[\frac{1}{\cos\left(\frac{\pi}{6}\right)}\right] = 1$$

$$= \frac{\sqrt{3}}{2} \times \frac{2}{\sqrt{3}} = 1$$

$\therefore \quad \cos 2x = 1$



3. a, b, c, are in AP

(i) $b + c, c + a, a + b$ are also in A.P.

a, b, c are in A.P. $\quad \Rightarrow \quad 2b = a + c$

$\Rightarrow \quad b - a = c - b \quad \Rightarrow \quad a - b = b - c$

Difference between term of given A.P. $= a - b, b - c$ which are equal by equation (i)

Hence $b + c, c + a, a + b$ is an A.P.

(ii) $\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab}$ are also in A.P.

Common difference $= \frac{(b-a)}{cab}, \frac{(c-b)}{abc}$

By equation (i) $b - a = c - b$ *i.e.,* difference between terms is same

Hence the given series is in A.P.

(iii) $a^2(b+c), b^2(c+a), c^2(a+b)$

$$\text{Difference} = \underbrace{b^2c + b^2a - a^2b - a^2c}_{d_1}, \underbrace{c^2a + c^2b - b^2c - b^2a}_{d_2}$$

$$d_1 = c(b^2 - a^2) + ab(b-a) = (ca + ab + cb)(b-a)$$

$$= (ca + ba + bc)(c-b) \qquad \text{[from eq. (i)]}$$

$$d_2 = a(c^2 - b^2) + bc(c-b) = (ac + ab + bc)(c-b)$$

$$d_1 = d_2$$

Hence given series is an A.P.

(iv) $a\left(\frac{1}{b} + \frac{1}{c}\right), b\left(\frac{1}{c} + \frac{1}{a}\right), c\left(\frac{1}{a} + \frac{1}{b}\right)$

$$\Rightarrow \quad \text{d1} = \frac{1}{c}(b-a) + \frac{b}{a} - \frac{a}{b}$$

$$\Rightarrow \quad d_2 = \frac{1}{a}(c-b) + \frac{c}{b} - \frac{b}{c}$$

$$\text{d1} = \frac{b-a}{c} + \frac{(b-a)(b+a)}{ab}$$

$$d_2 = \frac{c-b}{a} + \frac{(c-b)(c+b)}{bc}$$

$$= (b-a)\left[\frac{1}{c} + \frac{1}{a} + \frac{1}{b}\right]$$

$$= (b-a)\left[\frac{1}{c} + \frac{1}{a} + \frac{1}{b}\right]$$

$$= (c-b)\left[\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right]$$

From Eqn. (i) $\Rightarrow d_1 = d_2$

Hence given series is also an A.P.

4. 2G.P.s

$$\frac{a_1}{1-r_1} = 1 \Rightarrow a_1 = 1-r_1$$

$$\frac{a_2}{1-r_2} = 1 \Rightarrow a_2 = 1-r_2$$

$$a_1r_1 = a_2r_2 \Rightarrow (1-r_1)r_1 = (1-r_2)r_2$$

$$\Rightarrow \quad r_1 - r_2 = (r_1-r_2)(r_1+r_2)$$

$$\Rightarrow \quad r_1 + r_2 = 1$$

$$a_1r_1^2 = \frac{1}{8} \Rightarrow (1-r_1)r_1^2 = \frac{1}{8}$$

$$\Rightarrow \quad (2r_1-1)(4r_1^2-2r_1-1) = 0$$

If $r_1 = \frac{1}{2}$ then $r_1 = r_2$

$$\Rightarrow \quad 4r_1^2 - 2r_1 - 1 = 0 \Rightarrow r_1 = \frac{1\pm\sqrt{5}}{4}$$

If $r_1 = \frac{1-\sqrt{5}}{4}$ then, $r_2 > 1$

$$\Rightarrow \quad r_1 = \frac{1+\sqrt{5}}{4}$$

$$\therefore \quad a_1r_1 = (1-r_1)r_1 = \left[1-\left(\frac{1+\sqrt{5}}{4}\right)\right]\left(\frac{1+\sqrt{5}}{4}\right)$$

$$= \left(\frac{3-\sqrt{5}}{4}\right)\left(\frac{1+\sqrt{5}}{4}\right)$$

$$= \frac{\sqrt{5}-1}{8} = \frac{\sqrt{m}-n}{p}$$

$\therefore$ 100 m + 10 n + p = 500 + 10 + 8 = 518

5.
$$S = \frac{1}{a_1a_2} + \frac{1}{a_2a_3} \dots \frac{1}{a_na_{n+1}}$$

$$= \frac{1}{a_2-a_1}\left[\frac{a_2-a_1}{a_1a_2} + \frac{a_3-a_2}{a_2a_3}\dots\right]$$

As $a_2 - a_1 = a_3 - a_2 = \dots a_{n+1} - a_n$

$$= \frac{1}{a_2-a_1}\left[\frac{1}{a_1}-\frac{1}{a_2}+\frac{1}{a_2}-\frac{1}{a_3}\dots+\frac{1}{a_n}-\frac{1}{a_{n+1}}\right]$$

$$= \frac{1}{a_2-a_1}\left[\frac{1}{a_1}-\frac{1}{a_{n+1}}\right]$$

$$= \frac{a_{n+1}-a_1}{(a_2-a_1)a_1a_{n+1}} = \frac{nd}{d(a_1a_{n+1})} = \frac{n}{a_1a_{n+1}}$$

$$S = \frac{n}{a_1a_{n+1}} = \frac{n}{a_1(a_1+nd)}$$

$$\lim_{n\to\infty} S = \lim_{n\to\infty} \frac{n}{a_1(a_1+nd)}$$

$$= \lim_{n\to\infty} \frac{1}{a_1\left(d+\frac{a_1}{n}\right)} = \frac{1}{a_1d}$$

6. (a,d) $S_n < \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\cdot\frac{1}{1+\frac{k}{n}+\left(\frac{k}{n}\right)^2}$

$$= \int_0^1 \frac{dx}{1+x+x^2} = \frac{\pi}{3\sqrt{3}}$$

Now, $T_n > \frac{\pi}{3\sqrt{3}}$ as $h\sum_{k=0}^{n-1} f(kh) > \int_0^1 f(x)\,dx > h\sum_{k=1}^{n} f(kh)$

7. We have $S_k = \dfrac{\frac{k-1}{k!}}{1-\frac{1}{k}} = \dfrac{1}{(k-1)!}$

Now, $(k^2-3k+1)S_k = \{(k-2)(k-1)-1\} \times S_k$

$$= \frac{1}{(k-3)!} - \frac{1}{(k-1)!}$$

$$\Rightarrow \quad \sum_{k=1}^{100} |(k^2-3k+1)S_k|$$

$$= 1+1+2-\left(\frac{1}{99!}+\frac{1}{98!}\right)$$

$$= 4 - \frac{100^2}{100!}$$

$$\Rightarrow \quad \frac{100^2}{100} + \sum_{k=1}^{100} |(k^2-3k+1)S_k| = 4$$

8. Let the three digit be a, ar, ar^2 then according to hypothesis

$$100a + 10ar + ar^2 + 792 = 100ar^2 + 10ar + a$$

$$\Rightarrow \quad a(r^2-1) = 8 \qquad \dots(1)$$

and $a, ar+2, ar^2$ are in A.P.

then $\quad 2(ar+2) = a + ar^2$

$$\Rightarrow \quad a(r^2-2r+1) = 4 \qquad \dots(2)$$

Dividing (1) by (2),

then $\quad \dfrac{a(r^2-1)}{a(r^2-2r+1)} = \dfrac{8}{4}$

$$\Rightarrow \quad \frac{(r+1)(r-1)}{(r-1)^2} = 2$$

$$\Rightarrow \quad \frac{r+1}{r-1} = 2$$

$\therefore$ r = 3 from (1), a = 1

Thus digits are 1, 3, 9 and so the required number is 931.

9. Let a be the first term, r be the common ratio and l be the last term of a G.P.

Now, $S = a + ar + ar^2 + \ldots + \frac{l}{r} + l$

$$S = \frac{lr - a}{r - 1} \quad \ldots(1) \text{ (Let } r > 1)$$

and $P = (a)(ar)(ar^2)\ldots\left(\frac{l}{r^2}\right)\left(\frac{l}{r}\right)(l)$

$$= (al)\left(ar \cdot \frac{l}{r}\right)\left(ar^2 \cdot \frac{l}{r^2}\right)\ldots \frac{n}{2} \text{ terms}$$

$$= (al)(al)(al)\ldots \frac{n}{2} \text{ terms}$$

$\therefore$ $P = (al)^{n/2}$

$$P^2 = (al)^n \quad \ldots(2)$$

and $R = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \ldots + \frac{1}{l}$

$\therefore$ Common ratio $\frac{1}{r} < 1$

then $R = \dfrac{\frac{1}{a} - \frac{1}{lr}}{1 - \frac{1}{r}} = \frac{(lr - a)}{alr} \cdot \frac{r}{(r-1)}$

$\therefore$ $R = \frac{(lr - a)}{al(r-1)} \quad \ldots(3)$

Dividing (1) by (3), we get

$$\frac{S}{R} = al$$

$\therefore$ $\left(\frac{S}{R}\right)^n = (al)^n \quad \ldots(4)$

From (2) and (4), we get

$$P^2 = \left(\frac{S}{R}\right)^n$$

10. $\because a_1, a_2, a_3, \ldots, a_n$ are in A.P. Let d be the common difference of this A.P.

Then $a_{i+2} - a_{i+1} = a_{i+1} - a_i = d$ (say) $\quad \ldots(1)$

$$\frac{a_i + a_{i+2}}{2} = a_{i+1} \Rightarrow \frac{a_{i+1}}{a_i + a_{i+2}} = \frac{1}{2}$$

$$S = \frac{1}{2}\sum_{i=1}^{n} a_i a_{i+2}$$

$$= \frac{1}{2}\sum_{i=1}^{n}(a_{i+1} - d)(a_{i+1} + d) = \frac{1}{2}\sum_{i=1}^{n}(a_{i+1}^2 - d^2)$$

$$= \frac{1}{2}\sum_{i=1}^{n}\{(a_1 - id)^2 - d^2\}$$

$$= \frac{1}{2}\sum_{i=1}^{n}\{a_1^2 + 2a_1 id + (i^2 - 1)d^2\}$$

$$= \frac{1}{2}a_1^2\sum_{i=1}^{n}1 + a_1 d\sum_{i=1}^{n} i + \frac{d^2}{2}\left[\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n}1\right]$$

$$= \frac{1}{2}na_1^2 + a_1 d\frac{n(n+1)}{2} + \frac{d^2}{2}\left[\frac{n(n+1)(2n+1)}{6} - n\right]$$

$$= \frac{n}{2}\left[a_1^2 + a_1 d(n+1) + \frac{(n-1)(2n+5)}{6}d^2\right]$$