OBJECTIVE MATHEMATICS

Volume 1

Descriptive Test Series

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CHAPTER-5 : THEORY OF EQUATIONS

UNIT TEST-1

1. If $a > b > 0$ and $a^3 + b^3 + 27\,ab = 729$ then the quadratic equation $ax^2 + bx - 9 = 0$ has roots $\alpha, \beta\ (\alpha < \beta)$. Find the value of $4\beta - a\alpha$.

2. Let α and β be roots of $x^2 - 6\,(t^2 - 2t + 2)\,x - 2 = 0$ with $a > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then find the minimum value of $\dfrac{a_{100} - 2a_{98}}{a_{99}}$ (where $t \in R$)

3. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 - Kx^3 + Kx^2 + Lx + M = 0$, where K, L and M are real numbers, then the minimum value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ is $-n$. Find the value of n.

4. Consider $y = \dfrac{2x}{1 + x^2}$, where x is real, then the range of expression $y^2 + y - 2$ is $[a, b]$. Find $b - 4a$.

5. If a, b, c are real numbers, $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is a root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always lies between α and β.

6. Let $-1 \leq p \leq 1$. Show that the equation $4x^3 - 3x - p = 0$ has a unique root in the interval $\left[\dfrac{1}{2}, 1\right]$ and identify it.

7. For real values of x, if the expression $\dfrac{(ax - b)\,(dx - c)}{(bx - a)\,(cx - d)}$ assumes all real values then $(a^2 - b^2)$ and $(c^2 - d^2)$ must have the same sign.

8. The real numbers x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in A.P. Find the intervals in which β and γ lie.

Hints and Solutions

1. $a > b > 0$

Simplify the given equation.

$$a^3 + b^3 + 27ab = 729$$

$\Rightarrow \quad a^3 + b^3 + (-9)^3 - 3ab\,(-9) = 0$

Using : $a^3 + b^3 + c^3 - 3abc$

$$= (a+b+c)\,(a^2 + b^2 + c^2 - ab - bc - ca)$$

$\Rightarrow (a + b - 9)\,(a^2 + b^2 - ab + 9a + 9b + 81) = 0$

Therefore, $a + b - 9 = 0$

$$a + b = 9$$

Let, $\quad f(x) = ax^2 + bx + c$

$$ax^2 + bx - 9 = 0 \Rightarrow \alpha + \beta = \frac{-b}{a}$$

$$\Rightarrow \quad \alpha\beta = \frac{-9}{a}\ (\alpha < \beta)$$

$$a > b > 0$$

$$f(1) = a + b - 9$$

Thus it is clear that 1 is the root of given quadratic equation.

either $a = 1$ or $b = 1$ if $b = 1$

$$\alpha = \frac{-9}{a}$$

We need $\quad 4\beta - a\alpha = 4 \times 1 - a\left(-\dfrac{9}{a}\right)$

$$= 4 + 9 = 13$$

2. $x^2 - 6\,(t^2 - 2t + 2)\,x - 2 = 0$

Roots are α, β

$\therefore \quad \alpha^2 - 6(t^2 - 2t + 2)\,\alpha - 2$

$$\alpha^2 - 2 = 6(t^2 - 2t + 2)\,\alpha$$

$\therefore \quad \alpha^{100} = 6(t^2 - 2t + 2)\,\alpha^{98}$

$$\alpha^{100} - 2\alpha^{98} = 6(t^2 - 2t + 2)\,\alpha^{99}$$

$$\therefore\ \beta^{100} - 2\beta^{98} = 6(t^2 - 2t + 2)\,\beta^{99}$$

$$a_x = \alpha^x - \beta^x$$

$$\frac{a_{100} - 2a_{98}}{a_{99}} = \frac{\alpha^{100} - 2\alpha^{98} - \beta^{100} + 2\beta^{98}}{\alpha^{99} - \beta^{99}}$$

$$= \frac{6(t^2 - 2t + 2)(\alpha^{99} - \beta^{99})}{(\alpha^{99} - \beta^{99})}$$

minimum value of $(t^2 - 2t + 2) = 1$

$\therefore$ so minimum value = 6

3. Given, $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 - Kx^3 + Kx^2 + Lx + M = 0$.

Then from the relation between the root and coefficient we get,

$$\Sigma\alpha = \alpha + \beta + \gamma + \delta = K \quad ...(1)$$

and $$\Sigma\,\alpha.\beta = K \quad ...(2)$$

$$\Sigma\,\alpha\cdot\beta\cdot\gamma = -L$$

and $$\alpha\cdot\beta\cdot\gamma\cdot\delta = M.$$

Now, $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\Sigma\alpha)^2 - 2(\Sigma\alpha\cdot\beta)$

$= K^2 - 2K$ [Using (1) and (2)]

$= K^2 - 2K + 1 - 1$

$= (K - 1)^2 - 1.$

We have minimum value of $(K - 1)^2$ is 0 as $(K-1)^2 \geq 0.$

So the minimum value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ is –1.

4. Given, $y = \dfrac{2x}{1+x^2}$

$\Rightarrow \quad yx^2 - 2x + y = 0$

Since $x \in R$, discriminant ≥ 0

$\Rightarrow \quad 2^2 - 4y^2 \geq 0$

$\Rightarrow \quad 1 - y^2 \geq 0 \Rightarrow y^2 - 1 \leq 0$

$\Rightarrow \quad (y-1)(y+1) \leq 0$

$\Rightarrow \quad y \in [-1, 1]$

$$g(y) = y^2 + y - 2$$

We need to find extremums of $g(y)$ on the interval $-1 \leq y \leq 1$

$$g'(y) = 2y + 1 = 0 \Rightarrow y = -\frac{1}{2}$$

$$g\left(-\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2} - 2 = -\frac{9}{4}$$

$$g(-1) = 1 - 1 - 2 = -2$$

$$g(1) = 1 + 1 - 2 = 0$$

Range of $y^2 + y - 2$ is $\left[0, -\dfrac{9}{4}\right]$

So $b = 0$ and $a = -\dfrac{9}{4}$ and hence $b - 4a = 9$

5. $$a^2\alpha^2 + b\alpha + c = 0 \quad ...(1)$$

and $$\alpha^2\beta^2 - b\beta - c = 0 \quad ...(2)$$

Let $f(x) = a^2x^2 + 2bx + 2c$

$$f(\alpha) = a^2\alpha^2 + 2(b\alpha + c)$$

$$= a^2\alpha^2 - 2a^2\alpha^2 = -a^2\alpha^2 = \text{–ive, by (1)}$$

$$f(\beta) = a^2\beta^2 + 2(b\beta + c)$$

$$= a^2\beta^2 + 2a^2\beta^2 = 3a^2\beta^2$$

$= +$ive, by (2)

Since $f(\alpha)$ and $f(\beta)$ are of opposite signs then we know from theory of equations that a root γ of the equation $f(x) = 0$ lies between α and β.

6. $f(x) = 4x^3 - 3x - p = 0$

$$f'(x) = 12x^3 - 3 = 12\left(x^2 - \frac{1}{4}\right)$$

$$= 12\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right) = +\text{ive for } x \geq \frac{1}{2}$$

Thus $f(x)$ is an increasing function for $x \geq \dfrac{1}{2}$

Now $f\left(\dfrac{1}{2}\right) = \dfrac{1}{2} - \dfrac{3}{2} - p = -1 - p = -$ ive

$f(1) = 4 - 3 - p = 1 - p$

$= +$ive $\quad \because -1 \leq p \leq 1$

Since $f(a)$ and $f(b)$ are of opposite signs there exists at least one or in general three rots of $f(x) = 0$ between a and b *i.e.*, between $\dfrac{1}{2}$ and 1. But $f(x)$ is *an* increasing function for $x \geq \dfrac{1}{2}$. Hence there exists only one root in $\left[\dfrac{1}{2}, 1\right]$. Put $x = \cos\theta$.

$\Rightarrow \quad 4\cos^3\theta - 3\cos\theta - p = 0$

or $\quad \cos 3\theta = p$ or $\theta = \dfrac{1}{3}\cos^{-1} p$

or $\quad \cos^{-1} x = \left(\dfrac{1}{3}\cos^{-1} p\right)$

or $\quad x = \cos\left(\dfrac{1}{3}\cos^{-1} p\right)$ where $-1 \leq p \leq 1.$

7. y = given expression

$\Rightarrow (ab - bcy)\,x^2 + (ac + bd)\,(y - 1)\,x + (bc - ady) = 0$

Since x is real $\Rightarrow \Delta \geq 0$

$(ac + bd)^2\,(y-1)^2 - 4(ad - bcy)(bc - ady) \geq 0$
$\forall\ y \in R$

or

$$(ac-bd)^2 y^2 + 2\{2(a^2d^2+b^2c^2)-(ac+bd)^2\}y + (ac-bd)^2 \geq 0 \;\forall\; y \in R$$

Above expression is to be +ive and its first term is +ive.

Hence $\Delta < 0$

$4\{2(a^2d^2 + b^2c^2) - (ac + bd)^2\}^2 - 4(ac - bd)^4 = -$ ive

Apply $L^2 - M^2 = (L + M)(L - M)$ and cancel 4.

or $[2a^2d^2 + 2b^2c^2 - (ac + bd)^2 - (ac - bd)^2]$

$[2a^2d^2 + 2b^2c^2 - (ac + bd)^2 + (ac - bd)^2] < 0$

or $[2a^2d^2 + 2b^2c^2 - 2a^2c^2 - 2b^2d^2]$

or $[2a^2d^2 + 2b^2c^2 - 4abcd] < 0$

Again cancel 2, the second factor is $(ab - bc)^2$ which is + ive and first factor is

$a^2(d^2 - c^2) + b^2(c^2 - d^2)$

or $(c^2 - d^2)(b^2 - a^2) = -(a^2 - b^2)(c^2 - d^2)$

Hence the required condition is

$-(a^2 - b^2)(c^2 - d^2)(ad - bc)^2 < 0$

or $(a^2 - b^2)(c^2 - d^2) > 0$ *i.e.*, +ive

Above will hold good if both $(a^2 - b^2)$ and $(c^2 - d^2)$ have the same sign *i.e.*, either both +ive and both –ive.

8. Since x_1, x_2, x_3 are in A.P., $2x^2 = x_1 + x_3$

$$\therefore \quad 3x_2 = \Sigma x_1 = 1$$

$$\therefore \quad x_2 = \frac{1}{3}$$

But x^3 is *a* root of given equation

$$\frac{1}{27} - \frac{1}{9} + \frac{1}{3}\beta + \gamma = 0$$

$$\text{or} \quad 9\beta + 27\gamma = 2 \text{ or } \beta + 3\gamma = \frac{2}{9} \quad \text{...(1)}$$

Again $\Sigma x_1x_2 - x_1x_2 + x_2x_3 + x_3x_1 = \beta$

and $\quad x_1x_2x_3 = -\gamma$

Putting $x_2 = \dfrac{1}{3}$, we get

$$\frac{1}{3}(x_1 + x_3) + x_1x_3 = \beta \text{ and } \frac{1}{3}x_1x_3 = -\gamma$$

Eliminating x_3 from the above relation

$$\frac{1}{3}\left(x_1 - \frac{3\gamma}{x_1}\right) - 3\gamma = \beta$$

$$x_1^2 - 3(\beta + 3\gamma)\, x_i - 3\gamma = 0$$

Since x_1 is real

$$\therefore \quad \Delta \geq 0$$

$$\therefore \quad 9(\beta + 3\gamma)^2 + 12\gamma \geq 0$$

$$\text{or} \quad 3(\beta + 3\gamma)^2 + 4\gamma \geq 0 \quad \text{...(2)}$$

We have now to find the intervals for β and γ by the help of (1) and (2).

$$\therefore \quad 3\left(\frac{2}{9}\right)^2 + 4\gamma \geq 0 \quad \text{by (1)}$$

$$\text{or} \quad \gamma + \frac{1}{27} \geq 0$$

$$\therefore \quad \gamma \geq -\frac{1}{27} \quad \text{...(3)}$$

and from (1)

$$9\beta + 27\gamma + 1 = 3$$

$$3 - 9\beta = 27\gamma + \geq 0, \text{ by (3)}$$

$$\text{or} \quad 3\beta - 1 \leq 0$$

$$\therefore \quad \beta \leq \frac{1}{3}.$$

$$\therefore \quad \beta \in \left]-\infty, \frac{1}{3}\right], \gamma \in \left]-\frac{1}{27}, \infty\right].$$