OBJECTIVE MATHEMATICS

Volume 1

Descriptive Test Series

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CHAPTER-14 : THE CIRCLE

UNIT TEST-1

1. The angle between a pair of tangents drawn from a point P to the circle $x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$ is 2α. Then find equation of the locus of the point P, is

2. α, β and γ are parametric angles of three point P, Q and R respectively on the circle $x^2 + y^2 = 1$ and A is the point $(-1, 0)$. If the lengths of the chords AP, AQ and AR are in G.P., then prove that $\cos\frac{\alpha}{2}, \cos\frac{\beta}{2}$ and $\cos\frac{\gamma}{2}$ are in G.P

3. Find the condition that chord of contact of any external point (h, k) to the circle $x^2 + y^2 = a^2$ should subtend right angle at the centre of the circle.

4. A circle in inscribed (i.e. touches all four sides) into a rhombous $ABCD$ with one angle 60°. The distance from the centre of the circle to the nearest vertex is equal to 1. If P is any point of the circle, then $|PA|^2 + |PB|^2 + |PC|^2 + |PD|^2$ is equal to:

5. Let x & y be the real number satisfying the equation $x^2 - 4x + y^2 + 3 = 0$. If the maximum and minimum values of $x^2 + y^2$ are M & m respectively, then find the numerical value of $(M + m)$.

6. Find number of values of 'c' for which the set, $\{(x, y) \mid x^2 + y^2 + 2x \le 1\} \cap \{(x, y) \mid x - y + c \ge 0\}$ contains only one point is common.

7. If (α, β) is a point on the circle whose centre is on the x-axis and which touches the line $x + y = 0$ at $(2, -2)$, then find the greatest value of 'α'.

8. The line $Ax + By + C = 0$, cuts the circle $x^2 + y^2 + ax + by + c = 0$ in P and Q and the line $A'x + B'y + C' = 0$ cuts the circle $x^2 + y^2 + a'x + b'y + c' = 0$ in R and S. If the four points P, Q, R, S are concylic, then

$$D = \begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$$

9. A variable circle passes through the point $A(a, b)$ & touches the x-axis and the locus of the other end of the diameter through A is $(x - a)^2 = \lambda by$, then find the value of λ

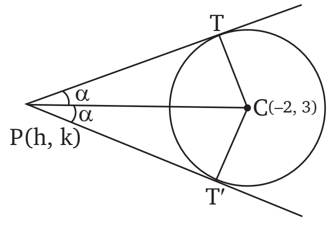
10. A circle touches the line $y = x$ at a point P such the $OP = 4\sqrt{2}$ where O is the origin. The circle contains the point $(-10, 2)$ in its interior and the length of its chord on the line $x + y = 0$ is $6\sqrt{2}$. Find the equation of the circle.

Hints and Solutions

1. The coordinates of the centre and radius of the given circle are $(-2, 3)$ and $\sqrt{4 + 9 - 9\sin^2\alpha - 13\cos^2\alpha} = 2\sin\alpha$ respectively.

Let the co-ordinates of CP be (h, k). Clearly, CP bisects

$$\angle TPT' = 2\alpha \quad \therefore \quad \angle CPT = \angle CPT' = \alpha$$



Now, in ΔCPT, we have $\sin\alpha = \dfrac{CT}{CP}$

$$\Rightarrow \sin\alpha = \frac{2\sin\alpha}{\sqrt{(h+2)^2+(k-2)^2}}$$

$$\Rightarrow (h+2)^2 + (k-3)^2 = 4$$

$$\Rightarrow h^2 + k^2 + 4h - 6k + 9 = 0$$

Hence, the locus of (h, k) is $x^2 + y^2 + 4x - 6y + 9 = 0$.

2. Co-ordinates of P, Q, R are $(\cos\alpha, \sin\alpha)$, $(\cos\beta, \sin\beta)$ and $(\cos\gamma, \sin\gamma)$ respectively. and $A \equiv (-1, 0)$

$$\therefore \quad AP = \sqrt{(1+\cos\alpha)^2+\sin^2\alpha} = 2\cos\frac{\alpha}{2}$$

$$AQ = \sqrt{(1+\cos\beta)^2+\sin^2\beta} = 2\cos\frac{\beta}{2}$$

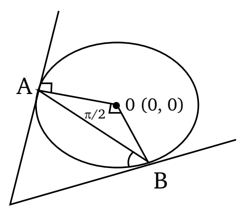
$$AR = \sqrt{(1+\cos\gamma)^2+\sin^2\gamma} = 2\cos\frac{\gamma}{2}$$

$\therefore$ AP, AQ, AR are in G.P., then $\cos\dfrac{\alpha}{2}, \cos\dfrac{\beta}{2}, \cos\dfrac{\gamma}{2}$ are also in G.P.

Hence (b) is the correct answer.

3. Equation of chord of contact AB is $hx + ky = a^2$ ...(1)

For equation of pair of tangents of OA and OB, make homogeneous $x^2 + y^2 = a^2$ with the help of



$$hx + ky = a^2$$

or $$\frac{hx+ky}{a^2} = 1$$

then $$x^2 + y^2 = a^2\left(\frac{hx+ky}{a^2}\right)^2$$

$$\Rightarrow \quad a^2(x^2 + y^2) = (hx + ky)^2$$

$$\Rightarrow \quad x^2(a^2 - h^2) - 2h\,kxy + y^2(a^2 - k^2) = 0$$

but $$\angle AOB = \frac{\pi}{2}$$

$\therefore$ Coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow a^2 - h^2 + a^2 - k^2 = 0$$

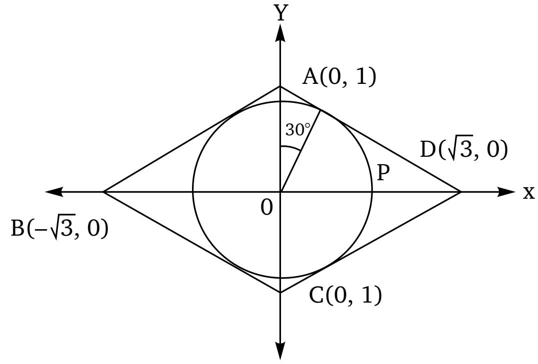
$$\Rightarrow \quad h^2 + k^2 = 2a^2$$

$\therefore$ The locus of (h, k) is $x^2 + y^2 = 2a^2$ which is a director circle.

4. From figure, we have

$$OA = 1$$

$$r = OA\cos 30^\circ = \frac{\sqrt{3}}{2}$$



Equation of circle is $x^2 + y^2 = \dfrac{3}{4}$

where the centre is $(0, 0)$ and radius is $\dfrac{\sqrt{3}}{2}$

$PA^2 + PB^2 + PC^2 + PD^2$

$$(x_1-0)^2+(y_1-1)^2+(x_1+\sqrt{3})+(y_1-0)^2+x_1^2+$$
$$(y_1+1)^2+(x_1-\sqrt{3})^2+(y_1-0)^2$$

$$4x_1^2+4y_1^2+8 = 4\left(x_1^2+y_1^2\right)+8$$

Since x_1 and y_1 lies on the circle thus putting the value of $x^2 + y^2 = \dfrac{3}{4}$ in the above equation we have

$$PA^2 + PB^2 + PC^2 + PD^2 = 4\times\frac{3}{4}+8$$

$$PA^2 + PB^2 + PC^2 + PD^2 = 11$$

5. $$x^2 - 4x + y^2 + 3 = 0$$

$$\Rightarrow x^2 - 4x + 4 + y^2 - 1 = 0$$

$$(x-2)^2 + y^2 = 1$$

represents a circle with centre at (2, 0) & radius = 1 units

$$\therefore \quad x^2 + y^2 \le 3$$

$$x^2 + y^2 \ge 1$$

$$\therefore \quad M = 3, m = 1$$

$$M = m = 4$$

6. (2) $x^2 + y^2 + 2x - 1 \le 0 \qquad x - y + c \ge 0$

To contains only one point in common the line should be a tangent to the circle

$$\Rightarrow x^2 + y^2 + 2x + 1 - 1 - 1 \le 0 \Rightarrow (x+1)^2 + y^2 \le 2$$

$$\text{Centre} = (-1, 0)$$

$$\text{Radius} = \sqrt{2}$$

$$(-1, 0) \quad x - y + c \ge 0$$

$$d = \left|\frac{-1+c}{\sqrt{2}}\right| = \sqrt{2}$$

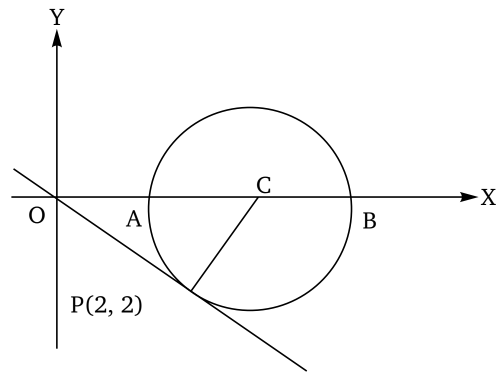
$$\Rightarrow \quad |c-1| = 2$$

$$c - 1 = 2 \quad c - 1 = -2$$
$$c = 3 \quad c = -1$$

7. Let $(a, 0)$ is the centre c and P is $(2, -2)$ which is given

Then $\angle COP = 45°$

Since the equation of OP is $x + y = 0$



thus also then $\angle OCP = 45°$

$\therefore \quad OP = 2\sqrt{2} = CP$

Hence, $\quad OC = 4$

The point on the circle with the greatest x co-ordinates is B.

$$a = OB = OC + CB = 4 + 2\sqrt{2}$$

8. Let the given circle be denoted by $S_1 = 0$ and $S_2 = 0$ and the points P, Q, R, S lie on the circle say $S_3 = 0$. PQ intersects both S_1 and S_3 and RS intersects both S_2 and S_3.

$\therefore$ PQ is radical axis of S_1 and S_3 and RS is radical axis of S_2 and S_3

$\therefore$ $Ax + By + C = 0$ is radical axis of S_1 and S_3

and $A'x + B'y + C' = 0$ is radical axis of S_2 and S_3

Also radical axis of S_1 and S_2 is given by

$$S_1 - S_2 = 0$$

or $(a - d')x + (b - b')y + (c - c') = 0$

Again we know that the radical axis of three circles taken in pairs are concurrent.

we have

$$\begin{vmatrix} a - a' & b - b' & c - c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$$

9. (d) Let the other end of diameter be (h, k)

$\therefore$ Equation of circle is

$$(x - a)(x - h) + (y - b)(y - k) = 0$$

$\therefore \quad$ Centre $\equiv \left(\frac{a+h}{2}, \frac{b+k}{2}\right)$

Since the circle touches the x-axis

$\therefore \quad |y\text{-co-ordinate}| = $ radius

$$\Rightarrow \quad \left|\frac{b+k}{2}\right| = \sqrt{\left(\frac{a+h}{2}\right)^2 + \left(\frac{b+k}{2}\right)^2 - (ah + bk)}$$

$$\therefore \quad \left(\frac{a+h}{2}\right)^2 = (ah + bk)$$

$\therefore$ Locus of point is

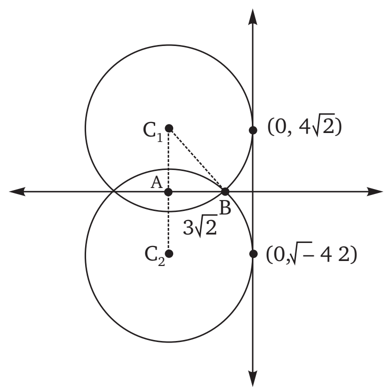
$$x^2 + 2ax + a^2 = 4ax + 4by$$
$$(x - a)^2 = 4by$$

10. Let the new co-ordinate axis be rotated by an angle of 45° in the clockwise direction. Then

$$X = x \cos(\theta) + y \sin(\theta)$$
$$Y = -x \sin(\theta) + y \cos(\theta)$$

where $\quad \theta = 45°$

$$\therefore \quad X = \frac{x - y}{\sqrt{2}}$$
$$Y = \frac{x + y}{\sqrt{2}}$$



In $\quad \Delta$ ABC, $AC = 4\sqrt{2}$

$$AB = 3\sqrt{2}$$

$$\therefore \quad \text{Radius} = \sqrt{\left(4\sqrt{2}\right)^2 + \left(3\sqrt{2}\right)^2} = b\sqrt{2}$$

$\therefore$ Equation of the circle is

$$\left(x + 5\sqrt{2}\right)^2 + \left(y + 4\sqrt{2}\right)^2 = \left(5\sqrt{2}\right)^2$$

or, $\left(\frac{x-y}{\sqrt{2}} + 5\sqrt{2}\right)^2 + \left(\frac{x+y}{\sqrt{2}} + 4\sqrt{2}\right)^2 = \left(5\sqrt{2}\right)$

or $(x - y + 10)^2 + (x + y \pm 8)^2 = 100$

But, since $(-10, 2)$ lies inside the circle. The equation of the circle is

$(x - y + 10)^2 + (x + y + 8)^2 = 100$

or $x^2 + y^2 + 100 - 2xy - 20y + 20x + x^2 + y^2 + 64 + 2xy + 16y + 16x = 100$

or, $2x^2 + 2y^2 + 36x - 4y + 64 = 0$

or, $x^2 + y^2 + 18x - 2y + 32 = 0$