CHAPTER 28

Definite Integrals

Exercise

1. $\int_{\pi/4}^{\pi/2} \cos\theta \operatorname{cosec}^2\theta \, d\theta$ is equal to

(a) $\sqrt{2}-1$ (b) $\sqrt{2}+1$

(c) $1+\frac{1}{\sqrt{2}}$ (d) $1-\frac{1}{\sqrt{2}}$

2. Consider the following statements.

(1) $\int_0^\infty \frac{1}{1+x^2} dx = \infty$ (2) $\int_{-\pi/2}^{\pi/2} \cos x \, dx = 2$

(3) $\int_{-\pi/2}^{\pi/2} \sin|x| \, dx = 2$ (4) $\int_{-\pi/2}^{\pi/2} \sin x \, dx = -2$

Which of these are correct?

(a) 1 and 2 (b) 2 and 4

(c) 3 and 1 (d) 2 and 3

3. $\int_{-2}^{2} x^3(1-x^2)\, dx$ is equal to

(a) $-\frac{40}{3}$ (b) $\frac{40}{3}$

(c) $\frac{5}{6}$ (d) 0

4. $\int_0^{\pi/2} \frac{\phi(x)}{\phi(x)+\phi(\pi/2-x)} dx$ is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{8}$ (d) 0

5. $\int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)} dx$ is equal to

(a) $\frac{\pi}{4}$ (b) a

(c) π (d) None of these

6. $\int_{\log 2}^{\log 4} \frac{dx}{\sqrt{e^x-1}}$ is equal to

(a) $\pi/6$ (b) $\pi/4$

(c) $\frac{\pi}{2}$ (d) None of these

7. If $I = \int_0^\pi f(\sin x)\, dx$, then $\int_0^\pi x\, f(\sin x)\, dx$ is

(a) $\left(\frac{\pi}{4}\right)I$ (b) $\left(\frac{\pi}{2}\right)I$

(c) πI (d) $\left(\frac{3\pi}{2}\right)I$

8. Find the value of $\int_{-a}^{a} \log\left(\frac{a-x}{a+x}\right) dx$

(a) $2a$ (b) a

(c) 0 (d) 1

9. Find the value of $\int_{1/\pi}^{2/\pi} \frac{\sin(1/x)}{x^2} dx$

(a) 1 (b) 1/2

(c) 3/2 (d) None of these

10. Find the value of $\int_0^\infty \frac{x\, dx}{(1+x)(1+x^2)}$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) π (d) None of these

11. If $A_n = \int_0^{\pi/4} \tan^n x\, dx, (n \ge 2)$, then $A_n + A_{n-2}$ is equal to

(a) $\frac{1}{n}+\frac{1}{n-1}$ (b) $\frac{1}{n+1}$

(c) $\frac{1}{n}$ (d) $\frac{1}{n-1}$

12. If $I_1 = \int_e^{e^2} \frac{dx}{\log x}$ and $I_2 = \int_1^2 \frac{e^x}{x} dx$, then

(a) $I_1 = I_2$ (b) $2I_1 = I_2$

(c) $I_1 + I_2 = 0$ (d) $I_1 = 2I_2$

13. $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x} dx$ is equal to

(a) 0 (b) $(a+b)\frac{\pi}{2}$

(c) $a + b$ (d) $(a+b)\frac{\pi}{4}$

14. If $\int_0^\infty \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}$ $= \frac{\pi}{2(a+b)(b+c)(c+a)}$, then $\int_0^\infty \frac{dx}{(x^2+4)(x^2+9)}$ is equal to

(a) $\frac{\pi}{60}$ (b) $\frac{\pi}{20}$

(c) $\frac{\pi}{40}$ (d) $\frac{\pi}{80}$

15. $\int_{-a}^{a} (1+x^3)^{-1} dx$ is equal to

(a) 0 (b) $2\int_0^a (1-x^6)^{-1} dx$

(c) $2\int_0^a (1+x^3)^{-1} dx$ (d) None of these

16. $\int_{-1}^{1} \sin^{-1} \frac{x}{1+x^2} dx$ is equal to

(a) $\pi/4$ (b) $\pi/2$

(c) $\pi/6$ (d) 0

17. If $\int_0^x f(t)\, dt = x + \int_x^1 t\, f(t)\, dt$, then $f(1)$ is equal to

(a) $\frac{1}{2}$ (b) 0

(c) 1 (d) $-\frac{1}{2}$

18. If $I_1 = \int_0^{3\pi} f(\cos^2 x)\, dx$ and $I_2 = \int_0^{\pi} f(\cos^2 x)\, dx$, then

(a) $I_1 = I_2$ (b) $I_1 = 2I_2$

(c) $I_1 = 5I_2$ (d) $I_1 = 3I_2$

19. If $\int_0^{\pi/2} \cos^m x \sin^m x\, dx = \lambda \int_0^{\pi/2} \sin^m x\, dx$, then λ is

(a) 2^m (b) 2^{-m}

(c) $\sqrt{2^m}$ (d) None of these

20. The value of $\int_0^{\pi/4} \log(1+\tan x)\, dx$ is

(a) $\frac{1}{2}\pi \log 8$ (b) $\frac{1}{8}\pi \log 2$

(c) $\frac{1}{4}\pi \log 2$ (d) None of these

21. $\int_0^1 |\sin 2\pi x|\, dx$ is equal to

(a) 0 (b) $-\frac{1}{\pi}$

(c) $\frac{1}{\pi}$ (d) $\frac{2}{\pi}$

22. If $f(x) + f(y) = f(x + y)$ and $\int_0^3 f(x)\, dx = \lambda$, then $\int_{-3}^{3} f(x)\, dx$ is

(a) $-2k$ (b) $2k$

(c) 0 (d) $k/2$

23. If $f(x)$ is a function satisfying $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$ for all non-zero x, then $\int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)\, dx$ equals

(a) $\sin\theta + \operatorname{cosec}\theta$ (b) $\sin^2\theta$

(c) $\operatorname{cosec}^2\theta$ (d) 0

24. If for every integer n, $\int_n^{n+1} f(x)\, dx = n^2$, then the value of $\int_{-2}^{4} f(x)\, dx$ is

(a) 16 (b) 14

(c) 19 (d) None of these

25. $f : R \to R$, $g : R \to R$ are continuous functions. The value of integral $\int_{-\pi/2}^{\pi/2} [f(x) + f(-x)][g(x) - g(-x)] dx$ is

(a) π (b) 1

(c) -1 (d) 0

26. $\frac{1}{c}\int_{ac}^{bc} f\left(\frac{x}{c}\right) dx$ is equal to

(a) $\frac{1}{c}\int_a^b f(x)\, dx$ (b) $\int_a^b f(x)\, dx$

(c) $c\int_a^b f(x)\, dx$ (d) $\int_{ac^2}^{bc^2} f(x)\, dx$

27. $\int_0^{3/2} [x^2]\, dx$ is equal to

(a) $2+\sqrt{2}$ (b) $2-\sqrt{2}$

(c) 3/2 (d) 3

28. $\int_a^b [|x-a| + |x-b|] dx$ is equal to

(a) $\frac{(b-a)^2}{2}$ (b) $\frac{b^2-a^2}{2}$

(c) $\frac{a^3-b^3}{2}$ (d) $(b-a)^2$

29. $\int_0^1 \tan^{-1} \frac{2x-1}{(1+x-x^2)} dx$ is equal to

(a) -1 (b) 1

(c) 0 (d) None of these

30. If $f(x) = ax^2 + bx + c$ such that $f(0) = 2, f'(0) = -3$, $f''(0) = 4$, then $\int_{-1}^{1} f(x)\, dx$ is

(a) -3 (b) 16/3
(c) 0 (d) None of these

ANSWERS

1.	(a)	**2.**	(d)	**3.**	(d)	**4.**	(a)	**5.**	(b)	**6.**	(a)	**7.**	(b)	**8.**	(c)	**9.**	(a)	**10.**	(a)
11.	(d)	**12.**	(a)	**13.**	(d)	**14.**	(a)	**15.**	(b)	**16.**	(d)	**17.**	(a)	**18.**	(d)	**19.**	(b)	**20.**	(b)
21.	(d)	**22.**	(c)	**23.**	(d)	**24.**	(c)	**25.**	(d)	**26.**	(b)	**27.**	(b)	**28.**	(d)	**29.**	(c)	**30.**	(b)

Explanations

1. (a) $\int_{\pi/4}^{\pi/2} \cos\theta \operatorname{cosec}^2\theta\, d\theta = \int_{\pi/4}^{\pi/2} \frac{\cos\theta}{\sin^2\theta} d\theta$

Let $\sin\theta = t$

$\Rightarrow \cos\theta\, d\theta = dt$

$$= \int_{1/\sqrt{2}}^{1} \frac{1}{t^2} dt = \left[-\frac{1}{t}\right]_{1/\sqrt{2}}^{1} = \sqrt{2} - 1$$

2. (d) (1) $\int_1^{\infty} \frac{1}{1+x^2} dx = [\tan^{-1} x]_0^{\infty} = \tan^{-1}\infty - \tan^{-1} 0$

$= \pi/2$

(2) $\int_{-\pi/2}^{\pi/2} \cos x\, dx = 2\int_0^{\pi/2} \cos x\, dx = 2[\sin x]_0^{\pi/2} = 2$

(3) $\int_{-\pi/2}^{\pi/2} \sin|x|\, dx = 2\int_0^{\pi/2} \sin|x|\, dx$

$= 2[\cos|x|]_0^{\pi/2} = 2$

(4) $\int_{-\pi/2}^{\pi/2} \sin x\, dx = 0$ {odd function}

3. (d) $f(x) = x^3(1 - x^2)$

$f(-x) = (-x)^3[1 - (-x)^2] = (-x)^3[1 - x^2]$

$\because\ f(-x) = -f(x)$ {odd function}

So, $\int_{-2}^{2} x^3(1 - x^2)\, dx = 0$

4. (a) $I = \int_0^{\pi/2} \frac{\phi(x)}{\phi(x) + \phi\left(\frac{\pi}{2} - x\right)} dx$ ...(i)

$I = \int_0^{\pi/2} \frac{\phi\left(\frac{\pi}{2} - x\right)}{\phi\left(\frac{\pi}{2} - x\right) + \phi(x)} dx$ ...(ii)

On adding eqs. (i) and (ii),

$2I = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}$

$I = \pi/4.$

5. (b) $I = \int_0^{2a} \frac{f(x)}{f(x) + f(2a - x)} dx$ ...(i)

$I = \int_0^{2a} \frac{f(2a - x)}{f(2a - x) + f(x)} dx$ ...(ii)

On adding eqs. (i) and (ii),

$2I = \int_0^{2a} dx = 2a$

$I = a$

6. (a) $I = \int_{\log 2}^{\log 4} \frac{e^{x/2}}{e^{x/2}\sqrt{(e^{x/2})^2 - 1}} dx$

Let $z = e^{x/2} \Rightarrow dz = \frac{1}{2} e^{x/2} dx$

$I = \int_{\sqrt{2}}^{2} \frac{dz}{|z|\sqrt{z^2 - 1}} = 2[\sec^{-1} z]_{\sqrt{2}}^{2}$

$= 2(\sec^{-1} 2 - \sec^{-1}\sqrt{2}) = 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{6}$

7. (b) Let $A = \int_0^{\pi} x\, f(\sin x)\, dx$ ...(i)

$A = \int_0^{\pi} (\pi - x) f\{\sin(\pi - x)\}\, dx$ ...(ii)

On adding eqs. (i) and (ii),

$2A = \int_0^{\pi} \pi\, f(\sin x)\, dx = \pi I$

So, $A = \frac{\pi I}{2}$

8. (c) $f(x) = \log\left(\frac{a - x}{a + x}\right)$

Now, $f(-x) = \log\left(\frac{a + x}{a - x}\right) = -\log\left(\frac{a - x}{a + x}\right)$

$\Rightarrow f(-x) = -f(x)$ {Odd function}

$\Rightarrow \int_{-a}^{a} f(x)\, dx = 0$

9. (a) Put $\frac{1}{x}=t$ and $\frac{1}{x^2}dx=-dt$

$$I=-\int_{\pi}^{\pi/2}\sin t\,dt=\int_{\pi/2}^{\pi}\sin t\,dt$$

$$=[-\cos t]_{\pi/2}^{\pi}=\left[-\cos\pi+\cos\frac{\pi}{2}\right]=1$$

10. (a) Put $x=\tan\theta$ $\therefore$ $dx=\sec^2\theta\,d\theta$

$$I=\int_0^{\pi/2}\frac{\tan\theta\sec^2\theta\,d\theta}{(1+\tan\theta)(\sec^2\theta)}$$

$$=\int_0^{\pi/2}\frac{\sin\theta\,d\theta}{\cos\theta+\sin\theta}=\frac{\pi}{4}$$

11. (d) $An=\int_0^{\pi/4}\tan^n x\,dx$

$$A_n+A_{n-2}=\int_0^{\pi/4}(\tan^n x+\tan^{n-2}x)dx$$

$$=\int_0^{\pi/4}\tan^{n-2}x(1+\tan^2 x)dx$$

$$=\int_0^{\pi/4}\sec^2 x\tan^{n-2}x\,dx$$

Let $\tan x=t \Rightarrow \sec^2 x\,dx=dt$

$$A_n+A_{n-2}=\int_0^1 t^{n-2}dt=\left[\frac{t^{n-1}}{n-1}\right]_0^1=\frac{1}{n-1}$$

12. (a) $I_1=\int_e^{e^2}\frac{dx}{\log x}$ and $I_2=\int_1^2\frac{e^x}{x}dx$

Put $\log x=t$ in I_1 then $I_1=\int_1^2\frac{e^t}{t}dt$

or $I_1=\int_1^2\frac{e^x}{x}dx=I_2$

13. (d) $I=\int_0^{\pi/2}\frac{a\sin x+b\cos x}{\sin x+\cos x}dx$ ...(i)

$$I=\int_0^{\pi/2}\frac{a\sin\left(\frac{\pi}{2}-x\right)+b\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx \quad \text{...(ii)}$$

On adding eqs. (i) and (ii),

$$2I=\int_0^{\pi/2}(a+b)dx=(a+b)\frac{\pi}{2}$$

$$I=(a+b)\frac{\pi}{4}$$

14. (a) Given, $\int_0^{\infty}\frac{x^2dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}$

$$=\frac{\pi}{2(a+b)(b+c)(c+a)}$$

Put $a=0, b=2, c=3$ in both sides

$$\Rightarrow \int_0^{\infty}\frac{dx}{(x^2+4)(x^2+9)}=\frac{\pi}{2(2)(5)(3)}=\frac{\pi}{60}$$

15. (b) $I=\int_{-a}^{a}\frac{dx}{1+x^3}=\int_{-a}^{0}\frac{dx}{1+x^3}+\int_0^a\frac{dx}{1+x^3}$

$$=-\int_a^0\frac{dz}{1-z^3}+\int_0^a\frac{dx}{1+x^3}$$

{Put $z=-x$ in first integral}

$$=\int_0^a\frac{dx}{1-x^3}+\int_0^a\frac{dx}{1+x^3}$$

$$=\int_0^a\left[\frac{1}{1-x^3}+\frac{1}{1+x^3}\right]dx$$

$$=2\int_0^a\frac{1}{1-x^6}dx=2\int_0^a(1-x^6)^{-1}dx$$

16. (d) $f(x)=\sin^{-1}\frac{x}{1+x^2}$

$f(-x)=-\sin^{-1}\frac{x}{1+x^2}=-f(x)$ {odd function}

So, $\int_{-1}^{1}\sin^{-1}\frac{x}{1+x^2}dx=0$

17. (a) Given, $\int_0^x f(t)dt=x+\int_x^1 t\,f(t)dt$ ...(i)

Diff. w. r. to x,

$$f(x)=1+\{0-xf(x)\}$$

$$\Rightarrow f(x)=\frac{1}{1+x}$$

$$\Rightarrow f(1)=\frac{1}{2}$$

18. (d) $I_1=\int_0^{3\pi}f(\cos^2 x)dx$

$\because$ $\cos^2(\pi+x)=\cos^2 x$

$$\Rightarrow I_1=3\int_0^{\pi}f(\cos^2 x)dx=3I_2$$

19. (b) $I=\int_0^{\pi/2}\frac{2^m\cos^m x\sin^m x\,dx}{2^m}$

$$I=2^{-m}\int_0^{\pi/2}(\sin 2x)^m dx$$

Let $2x=t$, then $I=2^{-m}\int_0^{\pi}\sin^m t\frac{dt}{2}$

$$I=2^{-m}\cdot 2\cdot\int_0^{\pi/2}\sin^m t\frac{dt}{2}$$

$$I=2^{-m}\int_0^{\pi/2}\sin^m x\,dx$$

So, $\lambda=2^{-m}$

20. (b) $I = \int_0^{\pi/4} \log(1 + \tan x)\, dx$ ...(i)

$= \int_0^{\pi/4} \log\left\{1 + \tan\left(\frac{\pi}{4} - x\right)\right\} dx$

$= \int_0^{\pi/4} \log\left(1 + \frac{1 - \tan x}{1 + \tan x}\right) dx$

$I = \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan x}\right) dx$ ...(ii)

From eqs. (i) and (ii),

$2I = \int_0^{\pi/4} \log 2 \Rightarrow 2I = \frac{\pi}{4}\log 2$

So, $I = \frac{\pi}{8}\log 2$

21. (d) $\int_0^1 |\sin 2\pi x|\, dx$

$= \int_0^{1/2} \sin 2\pi x\, dx + \int_{1/2}^1 (-\sin 2\pi x)\, dx$

$= \left[-\frac{\cos 2\pi x}{2\pi}\right]_0^{1/2} + \left[\frac{\cos 2\pi x}{2\pi}\right]_{1/2}^1 = \frac{2}{\pi}$

22. (c) $f(x) + f(y) = f(x + y)$

Put $x = 0$ and $y = 0$, then $f(0) = 0$

Put $y = -x$, then $f(x) + f(-x) = f(0)$

$\Rightarrow$ $f(-x) = -f(x)$ odd function

So, $\int_{-3}^{3} f(x)\, dx = 0$

23. (d) $x^2 f(x) + f\left(\frac{1}{x}\right) = 0$

$\Rightarrow \quad f(x) = -\frac{1}{x^2} f\left(\frac{1}{x}\right)$ ...(i)

$I = \int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)\, dx$

Let $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2} dt$

then $I = \int_{\operatorname{cosec}\theta}^{\sin\theta} -\frac{1}{t^2} f\left(\frac{1}{t}\right) dt$

$= \int_{\operatorname{cosec}\theta}^{\sin\theta} f(t)\, dt$ {From eq. (i)}

$I = -\int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)\, dx = -I$

or $2I = 0 \Rightarrow I = 0$

24. (c) n and $n + 1$ are consecutive integers.

$\int_n^{n+1} f(x)\, dx = n^2$

So,

$\int_{-2}^{4} f(x)\, dx = \int_{-2}^{-1} f(x)\, dx + \int_{-1}^{0} f(x)\, dx + \int_0^1 f(x)\, dx$

$+ \int_1^2 f(x)\, dx + \int_2^3 f(x)\, dx + \int_3^4 f(x)\, dx$

$= (-2)^2 + (-1)^2 + (0)^2 + (1)^2 + (2)^2 + (3)^2 = 19$

25. (d) Let $F(x) = [f(x) + f(-x)][g(x) - g(-x)]$

$= f(x)\, g(x) - f(x)\, g(-x) + f(-x)\, g(x) - f(-x)\, g(-x)$

$F(-x) = f(-x)\, g(-x) - f(-x)\, g(x) + f(x)\, g(-x) - f(x)\, g(x)$

$= -F(x)$

i.e., odd function

$\int_{-\pi/2}^{\pi/2} [f(x) + f(-x)][g(x) - g(-x)]\, dx = 0$

26. (b) $I = \frac{1}{c}\int_{ac}^{bc} f\left(\frac{x}{c}\right) dx$

Let $\frac{x}{c} = t$

$\Rightarrow \quad I = \frac{1}{c}\int_a^b c\, f(t)\, dt = \int_a^b f(t)\, dt = \int_a^b f(x)\, dx$

27. (b) $x^2 = 0$ at $x = 0$

and $x^2 = \frac{9}{4}$ at $x = \frac{3}{2}$

In between 0 to $\frac{9}{4}$ there are three integers 0, 1, 2.

Also $x^2 = 0, 1, 2 \Rightarrow x = 0, 1, \sqrt{2}$

$I = \int_0^{3/2} [x^2]\, dx$

$= \int_0^1 0\, dx + \int_1^{\sqrt{2}} 1\, dx + \int_{\sqrt{2}}^{3/2} 2\, dx$

$= 0 + (\sqrt{2} - 1) + 2\left(\frac{3}{2} - \sqrt{2}\right)$

$= \sqrt{2} - 1 + 3 - 2\sqrt{2} = 2 - \sqrt{2}$

28. (d) $\because a < x < b$

So, $x - a = (+)$ve and $x - b = (-)$ve

$I = \int_a^b \{|x - a| + |x - b|\} dx$

$= \int_a^b \{(x - a) - (x - b)\} dx$

$= [(b - a)x]_a^b = (b - a)^2$

29. (c) $\int_0^1 \tan^{-1}\frac{2x - 1}{1 + x - x^2} dx$

$= \int_0^1 \tan^{-1}\left(\frac{x + x - 1}{1 - x(x - 1)}\right) dx$

$= \int_0^1 \{\tan^{-1} x + \tan^{-1}(x - 1)\}\, dx$

$I = I_1 + I_2$

where, $I_1 = \int_0^1 \tan^{-1} x\, dx$

and $I_2 = \int_0^1 \tan^{-1}(x-1)\,dx$

Now, $I_2 = \int_0^1 \tan^{-1}(x-1)\,dx$

$= \int_0^1 \tan^{-1}(1-x-1)\,dx = \int_0^1 \tan^{-1}(-x)\,dx$

$= -\int_0^1 \tan^{-1}(x)\,dx = -I_1$

So, $I_1 + I_2 = 0$

Hence, $I = 0$

30. (b) $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

$\because \quad f'(0) = -3 \Rightarrow b = -3$

$f''(x) = 2a$

$\because \quad f''(0) = 4 \Rightarrow a = 2$

and $f(0) = 2 \Rightarrow c = 2$

So, $f(x) = 2x^2 - 3x + 2$

$$\int_{-1}^{1} (2x^2 - 3x + 2)\,dx$$

$$= \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 2x\right]_{-1}^{1} = \frac{16}{3}$$